Demonstration.—Put OP_m to represent the m^{th} of the n roots of OP, then if the proposition is true $OP_m^n = OP$.

By prop. 1 $OP_m^n = OR$ revolved through $n(\frac{1}{n}\theta + \frac{1}{n}(m - 1)C$,

or $OP_m^n = OR$ " " θ,

since complete circumferences may be rejected.

But $OP = OR$ revolved through θ, therefore $OP_m^n = OP$.

PROPOSITION 4.—$OR^{\frac{1}{n}} = OR$ revolved through $\frac{1}{n}$ C.

Hence when $OP = OR^{\frac{1}{n}}$, OP is the n^{th} root of $+ 1$, or OR.

Demonstration.—When $\theta = RP$ becomes C, OP becomes OR, hence by substitution $OP^{\frac{1}{n}} = OR$ revolved through $\frac{1}{n}$ C (prop. 2),

becomes $OR^{\frac{1}{n}} = OR$ " " $\frac{1}{n}$ C.

This formula gives but one of the n roots of $+ 1$, or OR, which root I call the first of the n roots of $+ 1$.

By prop, 3 the m^{th} of the $n - 1$ remaining roots of $+ 1$, is the m^{th} power of the n^{th} root of $+ 1$.

It is plain that the n roots of $+ 1$ are *numerically* equal to each other but *geometrically* or *positionally* different.

The reader may test the above propositions by solving the binomial equations $x^2 = 1$, $x^3 = 1$, $x^4 = 1$, $x^5 = 1$, $x^6 = 1$, $x^8 = 1$, $x^{10} = 1$, $x^{12} = 1$, $x^{20} = 1$, etc., and constructing the roots in accordance with the convention that $\sqrt{-1}$ denotes *perpendicularity*. See my first article in the January ANALYST The passage from $OR = 1$ to $OR = a$ is simple. The next article will deal with imaginary exponents.

SOLUTION OF AN INDETERMINATE PROBLEM.

BY GEO. R. PERKINS, L L. D., UTICA, N. Y,

No. 5.—Find three right triangles having equal perimeters, and whose areas shall be in arithmetical progression.

Since these triangles have the same perimeters, and their areas are in arithmetical progression, it is evident that the radii of their inscribed circles must be in arithmetical progression. We will therefore assume as follows:

First triangle. $\begin{cases} r + s & = \text{radius of inscribed circle.} \\ x + (r + s) & = \text{the base.} \\ y + (r + s) & = \text{the perpendicular.} \end{cases}$

Second triangle. $\begin{cases} r & = \text{radius of inscribed circle.} \\ x' + r & = \text{the base.} \\ y' + r & = \text{the perpendicular.} \end{cases}$

Third triangle. $\begin{cases} r - s & = \text{radius of inscribed circle.} \\ x'' + (r - s) & = \text{the base.} \\ y'' + (r - s) & = \text{the perpendicular.} \end{cases}$

Denoting the common perimeter of the three triangles by p, we have for their hypothenuses as follows:

$$x + y = \tfrac{1}{2}p - (r + s \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$x' + y' = \tfrac{1}{2}p - r \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$x'' + y'' = \tfrac{1}{2}p - (r - s)\ldots\ldots\ldots\ldots\ldots(3).$$

These conditions give $(x + y) + (x'' + y'') = 2(x' + y')$, which proves the hypothenuses to be in arithmetical progression.

Since the triangles are right, we have

$$[x + (r + s)]^2 + [y + (r + s)]^2 = (x + y)^2 \ldots\ldots\ldots(4),$$
$$[x' + r]^2 \qquad + [y' + r]^2 \qquad = (x' + y')^2 \ldots\ldots\ldots(5),$$
$$[x'' + (r - s)]^2 + [y'' + (r - s)]^2 = (x'' + y'')^2 \ldots\ldots(6).$$

These, when simplified, become

$$xy = (r + s)(x + y) + (r + s)^2 \ldots\ldots\ldots\ldots\ldots(7),$$
$$x'y' = r(x' + y') + r^2 \qquad \ldots\ldots\ldots\ldots\ldots(8),$$
$$x''y'' = (r - s)(x'' + y'') + (r - s)^2 \ldots\ldots\ldots(9).$$

Using the values of (1), (2) and (3), we have

$$xy = \tfrac{1}{2}p(r + s) \ldots\ldots\ldots\ldots\ldots(10),$$
$$x'y' = \tfrac{1}{2}pr \qquad \ldots\ldots\ldots\ldots\ldots(11),$$
$$x''y'' = \tfrac{1}{2}p(r - s) \ldots\ldots\ldots\ldots(12).$$

If from the squares of (1), (2) and (3) we subtract four times these products, and extract the square roots, we find

$$x - y = \sqrt{\tfrac{1}{4}p^2 - 3p(r + s) + (r + s)^2} \ldots\ldots\ldots\ldots\ldots(13),$$
$$x' - y' = \sqrt{\tfrac{1}{4}p^2 - 3pr + r^2} \qquad \ldots\ldots\ldots\ldots\ldots(14),$$
$$x'' - y'' = \sqrt{\tfrac{1}{4}p^2 - 3p(r - s) + (r - s)^2} \ldots\ldots\ldots\ldots(15).$$

Hence, we must have

$$\tfrac{1}{4}p^2 - 3p(r+s) + (r+s)^2 = \square \ldots\ldots\ldots\ldots (16),$$
$$\tfrac{1}{4}p^2 - 3pr + r^2 = \square \ldots\ldots\ldots\ldots (17),$$
$$\tfrac{1}{4}p^2 - 3p(r-s) + (r-s)^2 = \square \ldots\ldots\ldots\ldots (18).$$

Condition (16) will be satisfied if we take $p = 12(r+s)$, and (17) and (18) will become

$$r^2 + 36rs + 36s^2 = \square \ldots\ldots\ldots\ldots (19),$$
$$r^2 + 70rs + 73s^2 = \square \ldots\ldots\ldots\ldots (20).$$

Conditions (19) and (20) will be satisfied, by taking $r = 9s$, and consequently $p = 120s$.

Taking $s = 1$, we immediately obtain

$$x + y = 50;\quad x' + y' = 51;\quad x'' + y'' = 52;$$
$$x - y = 10;\quad x' - y' = 21;\quad x'' - y'' = 28.$$

Consequently, $x = 30$; $y = 20$;
$x' = 36$; $y' = 15$;
$x'' = 40$; $y'' = 12$.

First triangle.
$$\begin{cases} x + r + s = 40 = \text{base.} \\ y + r + s = 30 = \text{perpendicular.} \\ x + y = 50 = \text{hypothenuse.} \\ 120 = \text{perimeter.} \\ 600 = \text{area.} \end{cases}$$

Second triangle.
$$\begin{cases} x' + r = 45 = \text{base.} \\ y' + r = 24 = \text{perpendicular.} \\ x' + y' = 51 = \text{hypothenuse.} \\ 120 = \text{perimeter.} \\ 540 = \text{area.} \end{cases}$$

Third triangle.
$$\begin{cases} x'' + r - s = 48 = \text{base.} \\ y'' + r - s = 20 = \text{perpendicular.} \\ x'' + y'' = 52 = \text{hypothenuse.} \\ 120 = \text{perimeter.} \\ 480 = \text{area.} \end{cases}$$

I presume these results are the least values which will satisfy our Problem.

REMARK.—This Problem is the same as Problem 103, on page 465

of *John D. Williams' Elementary Treatise on Algebra*, Boston, 1840. The solution there given is satisfactory, but extremely lengthy, and very complex. The results obtained are as follows:

The sides of the first triangle	18601944; 13951458; 23252430.
" " " " second triangle	13999464; 18559223; 23247145.
" " " " third triangle	18515584; 14048388; 23241860.

Common perimeter = 55805832.

Radii of the inscribed circles are 4650486; 4655771; 4661056.

SOLUTION OF A PROBLEM.

BY PROF. DAVID TROWBRIDGE, WATERBURGH, N. Y.

Required the Sum of the Products of n Quantities taken m and m together, no two products having the same Factors.

Let the n quantities be $r_1, r_2, r_3, \ldots r_n$. Let S_m be the required sum. Also let

$$S_1 = R_1 = r_1 + r_2 + \ldots + r_n,\ R_2 = r_1^2 + r_2^2 + \ldots + r_n^2, \ldots$$
$$R_m = r_1^m + r_2^m + \ldots + r_n^m \ (1).$$

Now

$$r_1(r_2 + r_3 + \ldots + r_n) + r_2(r_1 + r_3 + \ldots + r_n) + \ldots$$
$$+ r_n(r_1 + r_2 + \ldots + r_{n-1}) = r_1(R_1 - r_1) + r_2(R - r_2) + \ldots$$
$$+ r_n(R_1 - r_n) = R_1^2 - R_2 = R_1S_1 - R_2.$$

But we have evidently taken each product twice, so that we shall consequently have

$$2S_2 = R_1S_1 - R_2 \ldots\ldots\ldots\ldots (2).$$

$$r_1(r_2r_3 + r_2r_4 + \ldots) + r_2(r_1r_3 + r_1r_4 + \ldots) + \ldots$$
$$+ r_n(r_1r_2 + r_1r_3 + \ldots) = r_1[S_2 - r_1(R_1 - r_1)]$$
$$+ r_2[S_2 - r_2(R_1 - r_2)] + \ldots + r_n[S_2 - r_n(R_1 - r_n)]$$
$$= R_1S_2 - R_1R_2 + R_3 = R_1S_2 - R_2S_1 + R_3.$$

But we have now taken each product three times. We hence have

$$3S_3 = R_1S_2 - R_2S_1 + R_3 \ldots\ldots\ldots\ldots (3).$$